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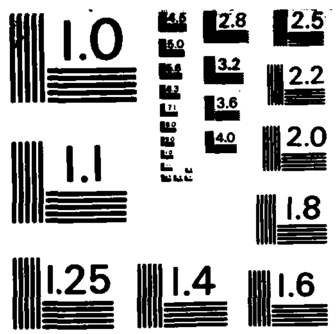

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The properties of a new rank statistic for comparing two populations in an arbitrary right censoring sampling scheme were investigated. The new statistic is an extension of the locally most powerful rank statistic for uncensored data and an extension of the well-known Wilcoxan statistic. The power of the new statistic was compared with the Prentic statistic for a sample size of IO and the new statistic was generally of slightly higher power for the logistic alternatives investigated. The investigators are optimistic that the statistic will remain more powerful against other alternatives.

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Introduction.

In the proposal, our goal was to study rank statistics to test the hypothesis H_o: $F^o(x)=G^o(x)$ for all x, where F^o and G^o denote the distribution functions of random variables X^o and Y^o respectively.

Let $X_1^o,\ldots,X_m^o$ and $Y_1^o,\ldots,Y_n^o$ be the two independent samples from F^o and G^o respectively, and $N=m+n$. Sample estimates of $F^o(x)$ and $G^o(x)$ are given by $F_N^o(x)=(\#\text{ of } X_i^o\leq x)/m$ and $G_N^o(x)=(\#\text{ of } Y_j^o\leq x)/n$ respectively, and to test H_o the locally most powerful linear rank statistic is given by

$$T_N = \int J(H_N^o(x))dF_N^o \;.$$

In the above statistic, $H_N^o = \frac{m}{N} F_N^o(x) + \frac{n}{N} G_N^o(x)$. The range of integration is $(-\infty,\infty)$ here and, unless otherwise specified, in the remainder of the report.

When X and Y observations are arbitrarly right censored, a better estimator of F(x) is given by the Kaplan-Meier product limit estimator. Thus, in the case of arbitrarly right censoring situation, it was proposed to study

$$T_N^* = \int J(H_N^*(x))dF_N^*(x) \;,$$

where F_N^* and H_N^* denote the product limit estimator of F and H respectively. Heuristic justifications for this statistic are

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that (i) it is a natural generalization of T_N, and (ii) a generalization of the Wilcoxon statistic, as obtained by Prentice (1978) using local optimum properties, contains product limit estimators. We had proposed to study asymptotic as well as small samples properties of T_N^*.

An important special case of T_N, obtained when $J(x)=x$, is known as the Wilcoxon statistic. If $J(x)=x$,

$$T_N = \int \{\frac{m}{N} F_N^o(x) + \frac{n}{N} G_N^o(x)\} dF_N^o(x) = \frac{m+1}{mN} + \frac{n}{N}\int G_N^o(x) dF_N^o(x) .$$

Since $(m+1)(mN)^{-1}$ is a constant, T_N is equivalent to

$$S_N = \int G_N^o(x) dF_N^o(x).$$

The statistic S_N is attractive for yet another reason that it estimates $Pr[Y^o \leq X^o]$. In the arbitrary right censoring situation, it would be appropriate to replace F_N^o and G_N^o by the Kaplan-Meier estimators F_N^* and G_N^* and therefore study

$$S_N^* = \int G_N^*(x) dF_N^*(x).$$

It turns out that Efron (1965) had obtained S_N^* as an extention of Gilbert (1962) and Gehan (1965) statistic.

In the first part of this report, Efron's statistic is compared with Prentice's statistic. In the second part, we consider some elementary properties of the proposed statistic, and in the third part, its small sample behavior is reported. It is observed that, unlike $H_N(x)$, $H_N^*(x)$ can not be expressed as

$$H_N^*(x) = \frac{m}{N} F_N^*(x) + \frac{n}{N} G_N^*(x).$$

Therefore, our statistic differs from Efron statistic. This aspect is elaborated in section 2 with its consequences.

Results of sections 2 and 3 are preliminary. Research is still continuing on some aspects of the proposed statistic. A full report will be published at the termination of such evaluations.

1. Efron and Prentice Statistics.

Let X^o and Y^o be two independent random variables with distribution functions (df) F^o and G^o respectively. Let U and V be two other random variables which are independent of each other, independent of X^o and Y^o, and with corresponding df I_U and I_V respectively. In the arbitraryly right censored data, the observables are given in terms of

$$X_i = \min (X_i^o, U_i) ,$$

$$\epsilon_i = \begin{cases} 1 \text{ if } X_i^o \le U_i , \\ 0 \text{ otherwise} , \end{cases}$$

where $i=1,\ldots,m$. Likewise, $Y_j=\min(Y_j^o,V_j)$ and $\epsilon_j=1$ (0) if $Y_j^o \le V_j$ ($Y_j^o > V_j$), $j=1,\ldots,n$ are observables from the other sample.

To compare F^o and G^o, Gehan (1965) and Gilbert (1962) independently proposed the following modification of the Wilcoxon statistic. They suggested the use of $\sum_i \sum_j W_G(i,j)$ where the "score function" $W_G(i,j)$ is defined as follows:

$$W_G(i,j) = \begin{cases} 1 & \text{if } x_i \ge y_j, \text{ when } y_j \text{ is an uncensored observation,} \\ 0 & \text{if } x_i < y_j, \text{ when } x_i \text{ is an uncensored observation,} \\ 1/2 & \text{if } x_i \text{ and } y_j \text{ are both censored.} \end{cases}$$

Thus, in Gehan statistic, a pair (x_i,y_j) contributes 1/2 whenever x_i^o and y_j^o "can not be compared". Efron (1965) proposed a modification to the above scoring function. He argued that the score function should be an estimate of $P[X^o \ge Y^o]$ and the estimate should be obtained conditional upon the available sample. This method of evaluating the score function has been shown to provide locally most powerful rank test under type II censoring by Bhattacharyya and Mehrotra (1983). Though the justification was obtained only recently, other occurences of this conditional argument have appeared in earlier works.

We consider an example of evaluation of Efron score. If x_i is a failure, y_j is a censored observation, and $y_j < x_i$, then the probability of the event $\{X^o \geq Y^o\}$ is given by $\{G^o(x_i) - G^o(y_j)\}/\{1-G^o(y_j)\}$. Clearly, the score associated with such a pair is

$$W_E(i,j) = \frac{G_N^*(x_i) - G_N^*(y_j)}{1-G_N^*(y_j)} ,$$

where G_N^* is the product limit estimator of the distribution function G^o. As in Gehan's case, $W_E(i,j)=1$ if $x_i \geq y_j$ and y_j is uncensored, 0 if $x_i < y_j$ and x_i is uncensored. The two scoring functions differ only when the pair (x_i, y_j) can not be compared. Efron has shown that his statistic, $T_E = \sum_i \sum_j W_E(i,j)$ can be represented as

$$T_E = \int G_N^*(x) dF_N^*(x) ,$$

where, as mentioned earlier, G_N^* and F_N^* are product limit estimators of G^o and F^o.

To obtain the asymptotic normality, one can apply the theory of U-statistic to the earlier form or Chernoff and Savage theorem to the integral representation of T_E. Briefly, the following result is obtained.

<u>Theorem (Efron):</u> Let m,n converge to infinity in such a way that lim $m/N=\lambda$, $0<\lambda<1$. Then, the distribution of

$$N^{1/2}\{T_E - \int (1-F^o)dG^o\}$$

converges to a normal distribution with mean zero and variance σ^2, which under the null hypothesis becomes

$$\sigma_o^2 = \frac{1}{4}\left[\frac{1}{\lambda}\int_o^1 \frac{z^3dz}{F\{F^{o-1}(z)\}} + \frac{1}{1-\lambda}\int_o^1 \frac{z^3dz}{G\{F^{o-1}(z)\}}\right].$$

Here $1-F=(1-F^o)(1-I_U)$ and $1-G=(1-F^o)(1-I_V)$.

From the above theorem, efficacy of Efron statistic is

$$\mathcal{E}_E = \frac{\{\int(1-F^o)dG^o\}^2}{\sigma_o^2}. \tag{1.1}$$

It should be remembered that the product limit estimator of the survival function $F^o=(1-F^o)$ is given by

$$\hat{P}(x) = \begin{cases} 1 & \text{if } x<x_{(1)}, \\ \prod_1^{k-1}\left(\frac{m-i}{m-i+1}\right)^{\delta_i} & \text{if } x_{(k)}\leq x<x_{(k+1)},\ k=1,\dots,m, \end{cases}$$

where $x_{(m+1)}=\infty$. Efron uses a slightly different estimator F_N^* of F^o. His estimator, which is self consistent i.e.

$$mF^*(s) = (\# \text{ of } x_i \geq s) + \sum_{x_i < s} (1-\delta_i)\frac{F^*(s)}{F^*(x_i)}$$

is identical to $\hat{P}(x)$ in all intervals $x_{(k)} \leq x < x_{(k+1)}$ except when $k=m$. In this last interval $F^*(s)=0$; i.e. it is assumed that $x_{(m)}$ is uncensored, irrespective of its actual value.

Using a conditional locally most powerful criterion, Prentice (1978) obtained a rank statistic T_p for testing H_o. His statistic is given by

$$T_p = \sum c_i\{Z_{(i)} + C_i \sum_j Z_{ij}\} ,$$

where the outer sum is over all failures, and the inner sum is over all those observations which are censored between the ith and the (i+1)th failures; $Z_{(i)}$ (and likewise Z_{ij}) takes value 1 if the ith failure (jth censored) is an X observation, and value 0 otherwise. The score c_i (C_i) corresponds to a failure (censored) observation. From the equivalence established in Mehrotra, Michalek, and Mihalko (1983), Prentice statistic can be written in a form whose asymptotic normality has been established by Shoenfeld (1982). Briefly, the efficacy of the Prentice statistic is given by

$$\mathcal{E}_p = \frac{[\int g(t)\log\{r_2(t)/r_1(t)\}\pi(t)\{1-\pi(t)\}V(t)dt]^2}{\int g^2(t)\pi(t)\{1-\pi(t)\}V(t)dt}, \qquad (1.2)$$

where

$g(t)=\lim(c_j-C_j)$, $r_1(t)=f^o(t)/\{1-F^o(t)\}$, $r_2(t)=g^o(t)/\{1-F^o(t)\}$,

$\pi(t)=(1-\lambda)\{1-G(t)\}/[\lambda\{1-F(t)\}+(1-\lambda)\{1-G(t)\}]$, and

$V(t)= f^{o}(t)\{1-I_U(t)\}+(1-\lambda)g^{o}(t)\{1-I_V(t)\}$.

The ratio of (1.1) and (1.2) gives the relative efficiency of Efron statistic versus Prentice statistic.

2. Properties of the Proposed Statistic.

The combined ranked sample of X's and Y's can be written in terms of three vectors $\underset{\sim}{W}, \underset{\sim}{\Delta}$, and $\underset{\sim}{Z}$, each with dimension N. The vector $\underset{\sim}{W}=(W_1,\ldots,W_N)$ represents the combined vector of ordered observations $W_1\leq\ldots\leq W_N$. $\underset{\sim}{\Delta}$ is a vector of indicator variables, where Δ_i takes value 1 if W_i is a failure, and 0 otherwise. $\underset{\sim}{Z}$ is another vector of indicator variables, with Z_i taking value 1 if W_i is an X observation and 0 otherwise. Using these notations, the product limit estimator of $\bar{H}^{o}(x)=mN^{-1}\bar{F}^{o}+nN^{-1}\bar{G}^{o}(x)$ at w_j is given by

$$\bar{H}_N^{*}(w_j) = \prod_{i=1}^{j}\left(\frac{N-i}{N-i+1}\right)^{\Delta_i} = \bar{H}_N^{*}(w_{j-1})\left(\frac{N-j}{N-j+1}\right)^{\Delta_j} .$$

The second factor on the right hand side of the above expression, can be expressed in terms of X and Y failures as

$$\frac{N-j}{N-j+1} = \frac{(m-\sum_{k=1}^{j}Z_k)+\{n-\sum_{k=1}^{j}(1-Z_k)\}}{(m-\sum_{k=1}^{j-1}Z_k)+\{n-\sum_{k=1}^{j-1}(1-Z_k)\}} . \tag{2.1}$$

By convention, $\sum_{k=1}^{0}Z_k=\sum_{k=1}^{0}(1-Z_k)=0$. On the other hand, the product limit estimators of the survival functions $\bar{F}^{o}$ and $\bar{G}^{o}$ are given by

$$F^*(w_j) = F^*(w_{j-1}) \left\{\frac{m-\sum_{k=1}^{j} Z_k}{m-\sum_{k=1}^{j-1} Z_k}\right\}^{\Lambda_j} , \tag{2.2}$$

and

$$\bar{G}^*(w_j) = \bar{G}^*(w_{j-1}) \left\{\frac{n-\sum_{k=1}^{j}(1-Z_k)}{n-\sum_{k=1}^{j-1}(1-Z_k)}\right\}^{\Lambda_j} . \tag{2.3}$$

From equations (2.1), (2.2), and (2.3), it is clear that, in the case of arbitrary right censoring,

$$H_N^* \neq \frac{m}{N} F_N^*(x) + \frac{n}{N} \bar{G}_N^*(x) .$$

As a consequence, the proposed statistic and Efron statistic differ from each other when $J(x)=x$. In other words, the proposed statistic is another generalization of the Wilcoxon statistic. Another interesting consequence of this observation is that, for $J(x)=x$, the statistic T_N^* does not estimate {constant+$\Pr[Y^o \leq X^o]$}. This appears to be a drawback of the proposed statistic. However, weights should be assigned according to the ranks in the combined sample. This justifies the usefulness of the proposed statistic, and consequently requires further investigation.

The mass function associated with the product limit estimator F_N^* can be shown to be

$$f_N^* = \{m- \frac{(1-\Lambda_i)Z_i}{F^*(w_i)}\}^{-1} ,$$

if w_k is an X observation and is uncensored. Hence, alternatively,

$$mf_N^*(w_k) = 1 + \sum_{i<k} \frac{Z_i(1-\Lambda_i)}{F^*(w_i)} f_N^*(w_k)$$

whenever $Z_k=1$ and $\Lambda_k=1$. The proposed statistic T_N can be expressed as

$$T_N^* = m^{-1} \sum_{\delta_k \Lambda_k = 1} J[H_N^*(w_k)]\{1 + \frac{Z_i(1- \ _i)}{F^*(w_i)} f_N^*(w_k)\} .$$

After changing the order of summation, the second term gives an interesting interpretation of T_N^*. Essentially, it amounts to assigning a contribution $H_N(w_k^*)$ at each X failure. From each censored observation that falls between two failures, the contribution is the weighted sum of the mass function $f_N^*(w_j)$; the weights are proportional to $H_N^*(w_j)$ and the summation is over all future X-censored observations. Prentice statistic is similar in nature and differs in the scores associated with the censored observations.

To obtain the asymptotic normality of the statistic, we consider the following representation of T_N^*:

$$T_N^* = \int J[H_N^*(x)]dF_N^*(x) = \int J[H^o(x)]dF^o(x) + \int J[H_N^*(x)-H^o(x)]dF^o(x)$$

$$+ \int J[H^O(x)]d[F_N^*(x)-F^o(x)] + \int \{J[H_N(x)]-J[H^o(x)]\}d[F_N^*(x)-F^o(x)].$$

The first term of the above expression is a constant, and under some regularity conditions on the behavior of J(x), the last term is asymptotically negligible. On the middle terms, one can apply

the property that $N^{1/2}\{\hat{H}^*(s)-H^o(s)\}$, considered as a stochastic process in s, approaches a normal process with zero mean and covariance kernel

$$H^o(s)H^o(t)\int_{-\infty}^{s}\frac{dH^o(s)}{\{1-H^o(z)\}\{1-H(z)\}}\ .$$

As a consequence, the two middle terms are asymptotically normally distributed with zero mean and appropriately obtained variance.

These and other related details are still under further investigations.

3. Small Sample Behavior of T_N^*.

At the present time, we have investigated the behavior of T_N^* for $J(x)=x$ which provides a generalized Wilcoxon statistic. We have compared this statistic with Prentice statistic which, in this section, will be denote by T_P. This comparison is made when the X and Y observations are generated from logistic populations. Prentice statistic is obtained with the appropriate "logistic" weights. Censoring varies over 0%, 10%, 30%, and 50%. The zero percent censoring is used to check the accuracy of the simulation results. Clearly, in this case, T_N and T_P are essentially the same, and both equivalent to the Wilcoxon statistic. In the alternative situation, the censored Y observations are generated by varying the location parameter, β, of the logistic

distribution from 0.1 to 0.9. In every case, the censoring distribution is a uniform whose range is chosen so that the desired censoring probability is attained. The sample sizes of the populations X and Y are both kept equal to 10. The power is obtained from 1000 repetitions. Table 1 shows 1000xpower of the statistics T_N^* and T_P.

TABLE 1: Simulated Powers of the Prentice and Proposed Statistics.

Prob. of Censor.	0.0		0.1		0.3		0.5	
β	T_P	T_N^*	T_P	T_N^*	T_P	T_N^*	T_P	T_N^*
0.0	50	50	50	50	50	50	50	50
0.1	56	56	60	65	47	57	55	76
0.2	59	59	81	82	55	69	65	80
0.3	84	84	96	106	65	78	71	79
0.4	109	109	115	121	99	119	107	113
0.5	148	148	151	165	136	134	133	151
0.6	167	167	179	195	147	171	147	156
0.7	196	196	185	190	175	173	160	146
0.8	225	225	236	230	189	188	192	179
0.9	292	292	285	296	225	227	180	190

Our simulation results show that the power of the proposed statistic T_N^* is generally larger than the power of T_P, though the difference is relatively small. This leads us to believe that T_N^* will continue to perform at least as well as Prentice statistic, even if the X's and Y's are generated from other distributions. Of course, the J function in T_N^* and the scores in the Prentice statistic must be chosen appropriately.

A simulation study, covering a wider range of distributions and sample sizes, is in progress. A technical report, based on this large study, is expected in the near future.

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